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The Hunter–Saxton equation describes the geodesic flow on a sphere

Jonatan Lenells

Department of Mathematics, University of California, Santa Barbara, CA 93106, USA

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Abstract

The Hunter–Saxton equation is the Euler equation for the geodesic flow on the quotient space $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}(\mathbb{S})$ of the infinite-dimensional group $\mathcal{D}(\mathbb{S})$ of orientation-preserving diffeomorphisms of the unit circle $\mathbb{S}$ modulo the subgroup of rotations $\mathrm{Rot}(\mathbb{S})$ equipped with the $\dot{H}^1$ right-invariant metric. We establish several properties of the Riemannian manifold $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}(\mathbb{S})$: it has constant curvature equal to 1, the Riemannian exponential map provides global normal coordinates, and there exists a unique length-minimizing geodesic joining any two points of the space. Moreover, we give explicit formulas for the Jacobi fields, we prove that the diameter of the manifold is exactly $\frac{\pi}{2}$, and we give exact estimates for how fast the geodesics spread apart. At the end, these results are given a geometric and intuitive understanding when an isometry from $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}(\mathbb{S})$ to an open subset of an L^2-sphere is constructed.

MSC: 35Q53; 58B20

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1. Introduction

The Hunter–Saxton equation

$$u_{txx} = -2u_x u_{xx} - u u_{xxx}, \quad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

models the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field, x being the space variable in a reference frame moving with the unperturbed wave speed and t being a slow time variable [7]. The liquid crystal state is a distinct phase of matter observed between the solid and liquid states. A nematic liquid crystal is characterized by long rigid molecules that have no positional order but tend to point in the same direction (along the director). In Eq. (1.1) $u(t, x)$ is a measure of the average orientation of the medium locally around x at time t (see [2] for a further discussion of the physical interpretation of (1.1)). Eq. (1.1) is a bi-variational, completely integrable system with a bi-Hamiltonian structure, implying the existence of an infinite family of commuting Hamiltonian flows together with an infinite sequence of conservation laws [8]. For spatially periodic functions, (1.1) describes the geodesic flow on the homogeneous space $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}(\mathbb{S})$ of the infinite-dimensional group $\mathcal{D}(\mathbb{S})$ of orientation-preserving diffeomorphisms of the unit circle $\mathbb{S}$ modulo the subgroup of rotations $\mathrm{Rot}(\mathbb{S})$, endowed

E-mail address: jonatan@math.ucsb.edu.

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with the $\dot{H}^1$ right-invariant metric given at the identity by $\langle u, v\rangle_{id} = \frac{1}{4}\int_{\mathbb{S}} u_x v_x \mathrm{d}x$ [9]. In fact, the Hunter–Saxton equation together with the well-known Korteweg–de Vries [15] and Camassa–Holm [3,4,14,16] equations describe all generic bi-Hamiltonian systems related to the Virasoro group (a one-dimensional extension of $\mathcal{D}(\mathbb{S})$) that can be integrated by the translation argument principle [9].

The hope is that the geometric interpretation of these nonlinear wave equations would provide additional insight into properties of their solutions. For example the rate at which geodesics spread apart is connected to the curvature of the underlying manifold, so that a positive sectional curvature would imply, at least formally, stability of the geodesic flow. However, in the cases of the Camassa–Holm and Korteweg–de Vries equations on the Virasoro group the sectional curvature does not have a definite sign [13,14], which makes it difficult to draw valuable conclusions. In this paper we prove that the situation for the Hunter–Saxton equation is quite different: the space $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}(\mathbb{S})$ endowed with the $\dot{H}^1$ right-invariant metric has constant positive sectional curvature. Moreover, it is shown that the Riemannian exponential map is a global diffeomorphism from a subset of the tangent space of the identity to all of $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}(\mathbb{S})$, providing global normal coordinates that are useful for the study of $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}(\mathbb{S})$. We prove that the diameter of the manifold is exactly $\frac{\pi}{2}$, and that there exist no conjugate points along any geodesics. This is related to another result: there exists a unique, globally length-minimizing, geodesic joining any two points of the manifold.

Two different choices for the space $\mathcal{D}(\mathbb{S})$ are possible: we may consider the Fréchet Lie group $\mathcal{D}^\infty(\mathbb{S})$ of smooth orientation-preserving diffeomorphism of $\mathbb{S}$, or the Banach manifold $\mathcal{D}^k(\mathbb{S})$ of orientation-preserving diffeomorphisms of the circle of Sobolev class H^k for $k \geq 3$. The Banach manifold structure of $\mathcal{D}^k(\mathbb{S})$ for $k \geq 3$ is more pleasant than the Fréchet manifold structure of $\mathcal{D}^\infty(\mathbb{S})$. On the other hand, the group operation on $\mathcal{D}^\infty(\mathbb{S})$ is smooth, whereas $\mathcal{D}^k(\mathbb{S})$ is only a topological group: right multiplication $R_\varphi : \psi \mapsto \psi \circ \varphi$ is smooth, but left multiplication $L_\psi : \varphi \mapsto \psi \circ \varphi$ is continuous but not C^1 (see [6]). For the sake of definiteness we choose in this paper to consider the case of $\mathcal{D}^k(\mathbb{S})$ with $k \geq 3$.

In Section 2 we review the Riemannian manifold structure of the space $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$ equipped with the $\dot{H}^1$ right-invariant metric. Its curvature is considered in Section 3. Explicit formulas for the Jacobi fields are given in Section 4, before the Riemannian exponential map is studied in Section 5. In Section 6 we show that the geodesics are globally length-minimizing, while the following section contains a construction of an isometry from $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$ to an open subset of the L^2-sphere of radius one. Finally, Section 8 contains some conclusions and remarks.

2. The Riemannian manifold $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$

We first need to review some properties of the space $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$ (see [12] for more details).

Let $k \geq 3$. Let $\mathcal{D}^k(\mathbb{S})$ denote the Banach manifold of orientation-preserving diffeomorphisms of $\mathbb{S}$ of Sobolev class H^k. By $\mathrm{Rot}(\mathbb{S}) \subset \mathcal{D}^k(\mathbb{S})$ we denote the subgroup of rotations $x \mapsto x + d$ for some $d \in \mathbb{R}$. Let $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$ be the space of right cosets $\mathrm{Rot}(\mathbb{S}) \circ \varphi = \{\varphi(\cdot) + d \mid d \in \mathbb{R}\}$ for $\varphi \in \mathcal{D}^k(\mathbb{S})$, and put $M^k = \{\varphi \in \mathcal{D}^k(\mathbb{S}) \mid \varphi(0) = 0\}$. Note that

$$M^k = \{u + \mathrm{i}d \mid u \in H^k(\mathbb{S}), u_x > -1, u(0) = 0\}, \tag{2.1}$$

so that M^k is an open subset of the closed hyperplane $\mathrm{i}d + \mathbf{E}^k \subset H^k(\mathbb{S})$, where $\mathbf{E}^k$ is the closed linear subspace $\mathbf{E}^k = \{u \in H^k(\mathbb{S}) \mid u(0) = 0\}$. The map $[\varphi] \mapsto \varphi - \varphi(0)$, where $[\varphi]$ denotes the equivalence class of φ in $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$, is a diffeomorphism $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S}) \to M^k$. In other words, M^k provides a global chart for $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$. In the sequel all the tangent spaces $T_\varphi M^k$, $\varphi \in M^k$, will be identified with $\mathbf{E}^k$ using this chart, albeit we normally keep the tangent space notation in order to make it clear where the different objects live.

Let $A = -D_x^2$ and let $\mathbf{F}^{k-2}$ be the subspace $\{f \in H^{k-2}(\mathbb{S}) \mid \int_{\mathbb{S}} f \mathrm{d}x = 0\}$. Then $A|_{\mathbf{E}^k}$ is an isomorphism $\mathbf{E}^k \to \mathbf{F}^{k-2}$. Let A^{-1} be its inverse given by

$$(A^{-1}f)(x) = -\int_0^x \int_0^y f(z)\mathrm{d}z\mathrm{d}y + x\int_{\mathbb{S}}\int_0^y f(z)\mathrm{d}z\mathrm{d}y, \quad f \in \mathbf{F}^{k-2}.$$

The $\dot{H}^1$ right-invariant metric is defined for two tangent vectors $U, V \in T_\varphi M^k \simeq \mathbf{E}^k$ at $\varphi \in M^k$ by

$$\langle U, V\rangle_\varphi = \frac{1}{4}\int_{\mathbb{S}} (U \circ \varphi^{-1}) A(V \circ \varphi^{-1})\mathrm{d}x = \frac{1}{4}\int_{\mathbb{S}} \frac{U_x V_x}{\varphi_x}\mathrm{d}x.$$

The reason for the introduction of the factor $\frac{1}{4}$ will become evident in Section 7. We write $|\cdot|_\varphi$ for the norm induced by $\langle\cdot,\cdot\rangle_\varphi$ on $T_\varphi M^k$, that is, $|U|_\varphi^2 = \frac{1}{4}\int_{\mathbb{S}} \frac{U_x^2}{\varphi_x}\mathrm{d}x$. There exists a covariant derivative ∇ compatible with the $\dot{H}^1$ right-invariant metric given in the global chart M^k by

$$(\nabla_X Y)(\varphi) = DY(\varphi)\cdot X(\varphi) - \Gamma(\varphi, Y(\varphi), X(\varphi)), \tag{2.2}$$

where $\Gamma : M^k \times \mathbf{E}^k \times \mathbf{E}^k \to \mathbf{E}^k$ is a smooth Christoffel map defined by

$$\Gamma(\varphi, U, V) = -\frac{1}{2}\left(A^{-1}D_x[(U\circ\varphi^{-1})_x(V\circ\varphi^{-1})_x]\right)\circ\varphi. \tag{2.3}$$

Note that Γ is right-invariant in the sense that

$$\Gamma(\varphi, U, V)\circ\psi = \Gamma(\varphi\circ\psi, U\circ\psi, V\circ\psi), \quad \varphi,\psi\in M^k,\ U, V\in\mathbf{E}^k. \tag{2.4}$$

The geodesics of the $\dot{H}^1$ right-invariant metric are described by Eq. (1.1). More precisely, let $J\subset\mathbb{R}$ be an open interval and let $\varphi : J\to\mathcal{D}^k(\mathbb{S})$ be a C^2-curve. Then the curve $u : J\to T_{id}\mathcal{D}^k(\mathbb{S})$ defined by

$$u : t\mapsto \varphi_t(t)\circ\varphi(t)^{-1}$$

satisfies the Hunter–Saxton equation

$$u_{txx} = -2u_x u_{xx} - u u_{xxx}, \tag{2.5}$$

if and only if the curve $[\varphi] : J\to \mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$ given by $[\varphi] : t\mapsto[\varphi(t)]$ is a geodesic with respect to ∇. The geodesics in $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})\simeq M^k$ can be found explicitly by the method of characteristics: for $u_0\in T_{id}M^k$ with $|u_0|_{id} = 1$ the unique geodesic $\varphi : [0, T^*(u_0))\to M^k$ with $\varphi(0) = id$ and $\varphi_t(0) = u_0$ is given by

$$\varphi(t) = \mathrm{i}d - \frac{1}{8}\left(A^{-1}D_x(u_{0x}^2)\right)(1-\cos 2t) + \frac{u_0}{2}\sin 2t,$$

where the maximal time of existence is

$$T^*(u_0) = \frac{\pi}{2} + \arctan\left(\frac{1}{2}\min_{x\in\mathbb{S}} u_{0x}(x)\right) < \frac{\pi}{2}. \tag{2.6}$$

For future reference we also note that

$$\sqrt{\varphi_x(t,x)} = \cos t + \frac{u_{0x}(x)}{2}\sin t, \quad t\in[0, T^*(u_0)),\ x\in\mathbb{S}, \tag{2.7}$$

and

$$(u_x\circ\varphi)(t,x) = 2\tan\left(\arctan\left(\frac{u_{0x}(x)}{2}\right) - t\right), \quad t\in[0, T^*(u_0)),\ x\in\mathbb{S}. \tag{2.8}$$

By definition a vector field $V : J\to TM^k$ along a C^2-curve $\varphi : J\to M^k$ is φ-parallel if $\nabla_{\varphi_t}V\equiv 0$. Define $u, v : J\to T_{id}M^k$ by

$$v(t) = V(t)\circ\varphi(t)^{-1}, \qquad u(t) = \varphi_t(t)\circ\varphi(t)^{-1}.$$

Then V is φ-parallel if and only if u and v solve the equation

$$v_{txx} = -\frac{3}{2}v_{xx}u_x - \frac{1}{2}v_x u_{xx} - v_{xxx}u, \quad t\in J,\ x\in\mathbb{S}. \tag{2.9}$$

3. Curvature

In this section we compute the curvature of the Riemannian manifold $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})\simeq M^k$ equipped with $\dot{H}^1$ right-invariant metric. In particular, we find the sectional curvature to be constant equal to 1. Recall that the curvature tensor R is defined for vector fields X, Y, Z on $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$ by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Theorem 3.1. *The curvature tensor of* $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S}) \simeq M^k$ *endowed with the* $\dot{H}^1$ *right-invariant metric is given for vector fields* X, Y, Z *by*

$$R\,(X,Y)\,Z = X\langle Y,Z\rangle - Y\langle X,Z\rangle. \tag{3.1}$$

In particular, $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$ *has constant sectional curvature equal to* 1.

Proof. In the chart M^k we have the following local formula for R in terms of the Christoffel map (cf. [10])

$$R(U,V)W = D_1\Gamma(\varphi,W,U)V - D_1\Gamma(\varphi,W,V)U + \Gamma(\varphi,\Gamma(\varphi,W,V),U) - \Gamma(\varphi,\Gamma(\varphi,W,U),V),$$
$$U,V,W \in T_\varphi M^k \simeq \mathbf{E}^k. \tag{3.2}$$

By the right-invariance (2.4) of Γ, it holds that $R(U,V)W = R(u,v)w$ if $U,V,W \in T_\varphi M^k$ and $u,v,w \in T_{id}M^k$ satisfy $u = U\circ\varphi^{-1}$, $v = V\circ\varphi^{-1}$, and $w = W\circ\varphi^{-1}$. Therefore, it is enough to consider the curvature at the identity $id \in M^k$.

Using (2.3) we compute, for $U,V,W \in T_\varphi M^k$,

$$\begin{aligned} D_1\Gamma(\varphi,W,U)V &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0}\Big(A^{-1}D_x\Big((W\circ(\varphi+\epsilon V)^{-1})_x(U\circ(\varphi+\epsilon V)^{-1})_x\Big)\Big)\circ(\varphi+\epsilon V)\\ &= \frac{1}{2}\Big(A^{-1}D_x\Big(((W\circ\varphi^{-1})_x V\circ\varphi^{-1})_x(U\circ\varphi^{-1})_x\Big)\Big)\circ\varphi\\ &\quad+\frac{1}{2}\Big(A^{-1}D_x\Big((W\circ\varphi^{-1})_x((U\circ\varphi^{-1})_x V\circ\varphi^{-1})_x\Big)\Big)\circ\varphi\\ &\quad-\frac{1}{2}\Big(A^{-1}D_x\Big((W\circ\varphi^{-1})_x(U\circ\varphi^{-1})_x\Big)\Big)_x\circ\varphi\cdot V, \end{aligned}$$

where we used that

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\bigg|_{\epsilon=0} U\circ(\varphi+\epsilon V)^{-1} = -(U\circ\varphi^{-1})_x V\circ\varphi^{-1}. \tag{3.3}$$

Therefore, for $u,v,w \in T_{id}M^k$,

$$D_1\Gamma(id,w,u)v = \frac{1}{2}A^{-1}D_x\left((w_x v)_x u_x\right) + \frac{1}{2}A^{-1}D_x\left(w_x(u_x v)_x\right) - \frac{1}{2}vD_xA^{-1}D_x\left(w_x u_x\right). \tag{3.4}$$

Moreover, for $f \in H^k(\mathbb{S})$,

$$D_xA^{-1}D_x f = -f + \int_{\mathbb{S}} f\,\mathrm{d}x. \tag{3.5}$$

Hence (3.4) may be simplified as

$$D_1\Gamma(id,w,u)v = \frac{1}{2}A^{-1}D_x\left(w_{xx}vu_x + 2u_xv_xw_x + w_xu_{xx}v\right) + \frac{1}{2}vw_xu_x - \frac{1}{2}v\int_{\mathbb{S}} w_xu_x\,\mathrm{d}x.$$

We arrive at

$$\begin{aligned} D_1\Gamma(id,w,u)v - D_1\Gamma(id,w,v)u &= \frac{1}{2}A^{-1}D_x\left(w_{xx}(vu_x - uv_x) + w_x(u_{xx}v - uv_{xx})\right)\\ &\quad+\frac{1}{2}w_x(vu_x - uv_x) - \frac{1}{2}v\int_{\mathbb{S}} w_xu_x\,\mathrm{d}x + \frac{1}{2}u\int_{\mathbb{S}} w_xv_x\,\mathrm{d}x. \end{aligned}$$

Furthermore, as $w_x(vu_x - uv_x) \in \mathbf{E}^k$,

$$A^{-1}D_x\left(w_{xx}(vu_x - uv_x) + w_x(u_{xx}v - uv_{xx})\right) = A^{-1}D_x^2\left(w_x(vu_x - uv_x)\right) = -w_x(vu_x - uv_x).$$

We get

$$D_1\Gamma(id,w,u)v - D_1\Gamma(id,w,v)u = -\frac{1}{2}v\int_{\mathbb{S}} w_x u_x \mathrm{d}x + \frac{1}{2}u\int_{\mathbb{S}} w_x v_x \mathrm{d}x. \tag{3.6}$$

On the other hand,

$$\begin{aligned}
&\Gamma(id,\Gamma(id,w,v),u) - \Gamma(id,\Gamma(id,w,u),v)\\
&= -\frac{1}{2}A^{-1}D_x\left(\left(-\frac{1}{2}A^{-1}D_x(w_x v_x)\right)_x u_x\right) + \frac{1}{2}A^{-1}D_x\left(\left(-\frac{1}{2}A^{-1}D_x(w_x u_x)\right)_x v_x\right)\\
&= \frac{1}{4}A^{-1}\left(\left(D_x^2 A^{-1}D_x(w_x v_x)\right)u_x\right) + \frac{1}{4}A^{-1}\left(\left(D_x A^{-1}D_x(w_x v_x)\right)u_{xx}\right)\\
&\quad - \frac{1}{4}A^{-1}\left(\left(D_x^2 A^{-1}D_x(w_x u_x)\right)v_x\right) - \frac{1}{4}A^{-1}\left(\left(D_x A^{-1}D_x(w_x u_x)\right)v_{xx}\right).
\end{aligned}$$

Using the identity $D_x^2 A^{-1} D_x = -D_x$ and (3.5), we rewrite this as

$$\begin{aligned}
&\frac{1}{4}A^{-1}\left(-(w_x v_x)_x u_x + (w_x u_x)_x v_x\right) + \frac{1}{4}A^{-1}\left(\left(-w_x v_x + \int_{\mathbb{S}} w_x v_x \mathrm{d}x\right)u_{xx}\right)\\
&\quad -\frac{1}{4}A^{-1}\left(\left(-w_x u_x + \int_{\mathbb{S}} w_x u_x \mathrm{d}x\right)v_{xx}\right).
\end{aligned}$$

Since $u, v \in \mathbf{E}^k$, we have $A^{-1}(u_{xx}) = -u$ and $A^{-1}(v_{xx}) = -v$. We obtain

$$\begin{aligned}
&\Gamma(id,\Gamma(id,w,v),u) - \Gamma(id,\Gamma(id,w,u),v)\\
&= \frac{1}{4}A^{-1}\left(-w_x v_{xx} u_x + w_x u_{xx} v_x\right) + \frac{1}{4}A^{-1}\left(-w_x v_x u_{xx} + w_x u_x v_{xx}\right) - \frac{1}{4}u\int_{\mathbb{S}} w_x v_x \mathrm{d}x + \frac{1}{4}v\int_{\mathbb{S}} w_x u_x \mathrm{d}x.
\end{aligned}$$

Thus

$$\Gamma(id,\Gamma(id,w,v),u) - \Gamma(id,\Gamma(id,w,u),v) = -\frac{1}{4}u\int_{\mathbb{S}} w_x v_x \mathrm{d}x + \frac{1}{4}v\int_{\mathbb{S}} w_x u_x \mathrm{d}x. \tag{3.7}$$

Adding (3.6) and (3.7) we conclude that

$$\begin{aligned}
R(u,v)w &= D_1\Gamma(id,w,u)v - D_1\Gamma(id,w,v)u + \Gamma(id,\Gamma(id,w,v),u) - \Gamma(id,\Gamma(id,w,u),v)\\
&= \frac{1}{4}\left(u\int_{\mathbb{S}} w_x v_x \mathrm{d}x - v\int_{\mathbb{S}} w_x u_x \mathrm{d}x\right) = u\langle v,w\rangle_{id} - v\langle u,w\rangle_{id}.
\end{aligned}$$

This completes the proof of formula (3.1).

To conclude that the sectional curvature is constant and equal to 1, we note that by definition

$$\mathrm{Sec}(X,Y) = \frac{\langle R(X,Y)Y, X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}.$$

It is straightforward to check that the expression (3.1) for R implies that this equals 1 for any vector fields X, Y. □

4. Jacobi fields

Our next task will be to compute the Jacobi fields along geodesics in $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S}) \simeq M^k$. Let $\varphi : [0, T^*(u_0)) \to M^k$ be a geodesic such that $\varphi(0) = id$ and $\varphi_t(0) = u_0$ for some element $u_0 \in T_{id}M^k$ with $|u_0|_{id} = 1$. Recall that a vector field $W : [0, T^*(u_0)) \to TM^k$ along φ is a Jacobi field if and only if it satisfies the Jacobi differential equation

$$\nabla_{\varphi_t}^2 W = R(\varphi_t, W)\varphi_t.$$

For $w_0 \in T_{id}M^k$ we let $t \mapsto W(t; w_0)$ be the unique Jacobi field along φ with $W(0; w_0) = 0$ and $\nabla_{\varphi_t} W(0; w_0) = w_0$. There is a unique way to write w_0 in the form $w_0 = c_0 v_0 + c_1 u_0$ with $c_0, c_1 \in \mathbb{R}$ and $v_0 \in T_{id}M^k$ orthogonal to u_0, i.e. $\langle v_0, u_0\rangle_{id} = 0$. Since M^k has constant curvature 1 we get (see [10] for further details)

$$W(t; w_0) = c_0 t\varphi_t(t) + c_1 V(t; v_0)\sin t, \tag{4.1}$$

where $t \mapsto V(t; v_0)$ is the parallel translation of v_0 along φ, i.e. V is the unique vector field along φ such that $V(0; v_0) = v_0$ and $\nabla_{\varphi_t} V \equiv 0$. We deduce that the first conjugate point of id along φ is $\varphi(\pi/2)$. However, formula (2.6) for the maximal existence time of φ says that $T^*(u_0) < \frac{\pi}{2}$, so we see that there are no points conjugate to id in M^k. By right-invariance of the metric we infer the following.

Proposition 4.1. *There are no conjugate points along any geodesics in* $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$. □

To find an explicit formula for the Jacobi field we need to find an expression for the parallel translation $t \mapsto V(t; v_0)$ in (4.1). This is the objective of the next lemma.

Lemma 4.2. *Let* $\varphi : J \to \mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S}) \simeq M^k$ *be the geodesic with* $\varphi(0) = id$ *and* $\varphi_t(0) = u_0$ *for* $u_0 \in T_{id}M^k$ *with* $|u_0|_{id} = 1$. *For* $v_0 \in T_{id}M^k$ *let* $t \mapsto V(t)$ *be the parallel translation of* v_0 *along* φ, *that is,* V *is the unique lift of* φ *to the tangent bundle such that* $V(0) = v_0$ *and* $\nabla_{\varphi_t} V \equiv 0$ *on* J. *Then*

$$V_x(t,x) = \sqrt{\varphi_x}\left(-2\langle u_0, v_0\rangle_{id}\left(\sin t + \frac{u_{0x}}{2}(1-\cos t)\right) + v_{0x}\right). \quad \square$$

Proof. Define $u, v : J \to T_{id}M^k$ by

$$v(t) = V(t) \circ \varphi(t)^{-1}, \qquad u(t) = \varphi_t(t) \circ \varphi(t)^{-1}.$$

By (2.9), u and v satisfy

$$v_{txx} = -\frac{3}{2}v_{xx}u_x - \frac{1}{2}v_x u_{xx} - v_{xxx}u, \quad t \in J,\ x \in \mathbb{S}.$$

Integrating with respect to x we obtain

$$v_{tx} = -v_{xx}u - \frac{1}{2}v_x u_x + d(t), \quad t \in J,\ x \in \mathbb{S}, \tag{4.2}$$

for some function $d : J \to \mathbb{R}$. To determine d we integrate both sides of (4.2) over $\mathbb{S}$ to find

$$d(t) = -\frac{1}{2}\int_{\mathbb{S}} v_x u_x \mathrm{d}x = -2\langle v, u\rangle_{id}.$$

Since V is parallel translated along φ, the inner product $\langle V, \varphi_t\rangle_\varphi$ is preserved along φ. By right-invariance of the metric $\langle v, u\rangle_{id} = \langle V, \varphi_t\rangle_\varphi$, so that $d(t) \equiv -2\langle v_0, u_0\rangle_{id} = d$ is independent of $t \in J$. We conclude that v and u solve the equation

$$v_{tx} = -v_{xx}u - \frac{1}{2}v_x u_x + d, \quad t \in J,\ x \in \mathbb{S}.$$

Since

$$(v_x \circ \varphi)_t = v_{tx} \circ \varphi + v_{xx} \circ \varphi \cdot \varphi_t = (v_{tx} + v_{xx}u) \circ \varphi$$

we find that

$$(v_x \circ \varphi)_t = -\frac{1}{2}v_x \circ \varphi \cdot u_x \circ \varphi + d.$$

Therefore, for a fixed $x \in \mathbb{S}$, $v_x \circ \varphi$ solves the ordinary differential equation

$$\dot{z}(t) = -\frac{1}{2}f(t)z(t) + d,$$

where, by (2.8),

$$f(t) = (u_x \circ \varphi)(t,x) = 2\tan\left(\arctan\left(\frac{u_{0x}(x)}{2}\right) - t\right).$$

The general solution is

$$z(t) = d\int_0^t \mathrm{e}^{(F(\tau)-F(t))/2}\mathrm{d}\tau + z(0)\mathrm{e}^{-F(t)/2}.$$

where $F(t) = \int_0^t f(s)\mathrm{d}s$. Now $\mathrm{e}^{F(t)/2}$ is the unique solution of $\left(\mathrm{e}^{F(t)/2}\right)_t = \frac{1}{2} f(t)\mathrm{e}^{F(t)/2}$ with $\mathrm{e}^{F(1)/2} = 1$. Since

$$(\sqrt{\varphi_x})_t = \frac{\varphi_{tx}}{2\sqrt{\varphi_x}} = \frac{u_x \circ \varphi}{2}\sqrt{\varphi_x},$$

we deduce by (2.7) that

$$\mathrm{e}^{F(t)/2} = \sqrt{\varphi_x} = \cos t + \frac{u_{0x}}{2}\sin t.$$

Using that $(v_x \circ \varphi)(0, x) = v_{0x}(x)$ we get

$$(v_x \circ \varphi)(t, x) = \frac{1}{\sqrt{\varphi_x}}\left(d\left(\sin t + \frac{u_{0x}}{2}(1 - \cos t)\right) + v_{0x}\right).$$

The proof is finished by observing that $V_x = v_x \circ \varphi \cdot \varphi_x$. □

We can now compute the Jacobi fields along geodesics.

Proposition 4.3. *Let $\varphi : [0, T^*(u_0)) \to M^k$ be the geodesic with $\varphi(0) = id$ and $\varphi_t(0) = u_0$ for some $u_0 \in T_{id}M^k$ with $|u_0|_{id} = 1$. Let w_0 be an element of $T_{id}M^k$ with decomposition $w_0 = c_0v_0 + c_1u_0$ for some $c_0, c_1 \in \mathbb{R}$, and $v_0 \in T_{id}M^k$ $\dot{H}^1$-orthogonal to u_0. The unique Jacobi field $t \mapsto W(t; w_0)$ with $W(0; w_0) = 0$ and $\nabla_{\varphi_t}W(0; w_0) = w_0$ is given by*

$$W(t; w_0) = c_0t\varphi_t(t) + \frac{c_1}{2}\left(v_0 \sin 2t + \int_0^x v_{0x}u_{0x}\mathrm{d}y \sin^2 t\right).$$

Proof. Since $\langle v_0, u_0\rangle_{id} = 0$, Lemma 4.2 implies that the parallel translation of v_0 along φ is given by

$$V(t; v_0) = \int_0^x v_{0x}(y)\left(\cos t + \frac{u_{0x}(y)}{2}\sin t\right)\mathrm{d}y.$$

The result now follows from the formula (4.1) for the Jacobi field. □

5. Riemannian exponential map

The Riemannian exponential map $\mathfrak{exp}_m$ at a point m of a Banach manifold $\mathcal{M}$ is always a local diffeomorphism at $0 \in T_m\mathcal{M}$ by the inverse mapping theorem. In the case of $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S}) \simeq M^k$ we will find that, for each ψ, $\mathfrak{exp}_\psi$ is in fact a global diffeomorphism, thereby providing global normal coordinates on the manifold.

Let U be a tangent vector at $\psi \in M^k$ and let $t \mapsto \varphi(t; U)$ be the geodesic with $\varphi(0; U) = \psi$ and $\varphi_t(0; U) = U$. Let $\mathfrak{D} \subset TM^k$ consist of all U such that the maximal existence time of $\varphi(\cdot; U)$ is larger than 1. Then $\mathfrak{exp}$ is the smooth map $\mathfrak{D} \to M^k$ defined by

$$\mathfrak{exp}(U) = \varphi(1; U).$$

The restriction of $\mathfrak{exp}$ to the tangent space $T_\psi M^k$ is denoted by $\mathfrak{exp}_\psi : \mathfrak{D}_\psi \to M^k$, where $\mathfrak{D}_\psi = \mathfrak{D} \cap T_\psi M^k$.

Theorem 5.1. *For any $\psi \in \mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}(\mathbb{S}) \simeq M^k$ the Riemannian exponential map is a diffeomorphism from $\mathfrak{D}_\psi \subset T_\psi M^k$ onto M^k.*

Proof. By right-invariance we may assume that $\psi = id$. We will first show that $\mathfrak{exp}_{id}$ is onto. Let ψ_1 be an arbitrary element in M^k. We need to find an element $u_0 \in T_{id}M^k$ with $|u_0|_{id} = 1$ and a time $t_1 > 0$ with $t_1 < T^*(u_0)$ such that $\mathfrak{exp}(t_1u_0) = \psi_1$. We may assume that $\psi_1 \neq id$. Then

$$0 < \int_{\mathbb{S}} \sqrt{\psi_{1x}}\mathrm{d}x < \left(\int_{\mathbb{S}} \psi_{1x}\mathrm{d}x\right)^{1/2} = 1,$$

so that we may choose $t_1 \in (0, \frac{\pi}{2})$ such that

$$\cos t_1 = \int_{\mathbb{S}} \sqrt{\psi_{1x}}\mathrm{d}x. \tag{5.1}$$

Let, for $x \in \mathbb{S} \simeq [0, 1]$,

$$u_0(x) = \frac{2}{\sin t_1} \left(\int_0^x \sqrt{\psi_{1x}} \mathrm{d}y - x \cos t_1 \right). \tag{5.2}$$

Note that by the choice of t_1 it holds that $u_0(0) = u_0(1) = 0$. As

$$u_{0x}(x) = \frac{2}{\sin t_1} \left(\sqrt{\psi_{1x}(x)} - \cos t_1 \right) \tag{5.3}$$

belongs to $H^{k-1}(\mathbb{S})$, we infer that $u_0 \in T_{id} M^k \simeq \mathbf{E}^k$. Moreover,

$$\int_{\mathbb{S}} u_{0x}^2 \mathrm{d}x = \frac{4}{\sin^2 t_1} \int_{\mathbb{S}} \left(\psi_{1x} - 2\sqrt{\psi_{1x}} \cos t_1 + \cos^2 t_1 \right) \mathrm{d}x.$$

Using that $\int_{\mathbb{S}} \psi_{1x} \mathrm{d}x = 1$ we find in view of (5.1) that

$$|u_0|_{id}^2 = \frac{1}{4} \int_{\mathbb{S}} u_{0x}^2 \mathrm{d}x = 1.$$

Furthermore, by (2.6) the condition $t_1 < T^*(u_0)$ amounts to

$$t_1 < \frac{\pi}{2} + \arctan \left(\frac{u_{0x}(x)}{2} \right) \quad \text{for } x \in \mathbb{S}.$$

This is equivalent to

$$2 \tan \left(t_1 - \frac{\pi}{2} \right) < u_{0x}(x) \quad \text{for } x \in \mathbb{S}.$$

Since $\tan \left(t_1 - \frac{\pi}{2} \right) = -\frac{1}{\tan t_1}$, formula (5.3) for u_{0x} shows that this holds if and only if

$$-\cos t_1 < \sqrt{\psi_x(x)} - \cos t_1 \quad \text{for } x \in \mathbb{S},$$

which is obviously a true statement. This proves that $t_1 < T^*(u_0)$.

Now let $\varphi : [0, T^*(u_0)) \to M^k$ be the geodesic with $\varphi(0) = id$ and $\varphi_t(0) = u_0$ so that $\mathfrak{exp}(t_1 u_0) = \varphi(t_1)$. Formula (2.7) yields

$$\sqrt{\varphi_x(t_1, x)} = \cos t_1 + \frac{u_{0x}}{2} \sin t_1.$$

Comparing this expression with (5.3) we find that

$$\varphi_x(t_1, x) = \psi_{1x}(x), \quad x \in \mathbb{S}.$$

Since $\varphi(t_1, 0) = \psi_1(0) = 0$, we obtain $\varphi(t_1) = \psi_1$. This proves that $\mathfrak{exp}$ maps the set $\mathfrak{D}_{id}$ onto M^k.

Let us now show that $\mathfrak{exp}_{id} : \mathfrak{D}_{id} \to M^k$ is injective. Suppose there exist $u_0, v_0 \in T_{id} M^k$ with $|u_0|_{id} = |v_0|_{id} = 1$ and times $t_1, s_1 > 0$ such that $t_1 < T^*(u_0)$, $s_1 < T^*(v_0)$, and

$$\mathfrak{exp}(t_1 u_0) = \mathfrak{exp}(s_1 v_0). \tag{5.4}$$

For $w \in T_{id} M^k$ let $\varphi(\cdot; w)$ be the geodesic with $\varphi(0; w) = id$ and $\varphi_t(0; w) = w$. We can phrase the assumption (5.4) as $\varphi(t_1; u_0) = \varphi(s_1; v_0)$. Thus (5.4) together with (2.7) give

$$\cos t_1 + \frac{u_{0x}}{2} \sin t_1 = \cos s_1 + \frac{v_{0x}}{2} \sin s_1. \tag{5.5}$$

Integrating both sides of this equation over $\mathbb{S}$ we deduce that $\cos t_1 = \cos s_1$. Since the maximal existence time is less than $\frac{\pi}{2}$ for all geodesics, we infer that $t_1 = s_1$. Now (5.5) clearly implies that $u_0 = v_0$. This establishes injectivity. Smoothness of the inverse of $\exp_{id}$ follows from the expressions (5.1) and (5.2). $\square$

From the proof of Theorem 5.1 we extract the following explicit formula for the logarithm.

Corollary 5.2. If $\psi \in M^k$ and we define $r \in (0, \frac{\pi}{2})$ by

$$r = \arccos\left(\int_{\mathbb{S}} \sqrt{\psi_x} \mathrm{d}x\right),$$

and $u_0 : \mathbb{S} \to \mathbb{R}$ by

$$u_0(x) = \frac{2}{\sin r}\left(\int_0^x \sqrt{\psi_x} \mathrm{d}y - x \int_{\mathbb{S}} \sqrt{\psi_x} \mathrm{d}x\right) = -\frac{2}{\sin r} A^{-1} D_x(\sqrt{\psi_x}),$$

then $ru_0 \in \mathfrak{D}_{id}$, $|u_0|_{id} = 1$, and

$$\mathfrak{exp}(ru_0) = \psi.$$

6. Length-minimizing geodesics

Using the global normal chart established in Theorem 5.1, we can immediately deduce the following theorem. In its statement $L(\alpha)$ denotes the length of a piecewise C^1-path α in $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$ with respect to the $\dot{H}^1$ right-invariant metric.

Theorem 6.1. *Any two points $\psi_0, \psi_1 \in \mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$ can be joined by a unique geodesic φ. Moreover, this geodesic is strictly length-minimizing. More precisely, if α is any piecewise C^1-path joining ψ_0 and ψ_1, then $L(\varphi) \le L(\alpha)$ and equality holds if and only if a reparametrization of α equals φ.*

Proof. The first half is obvious in the light of Theorem 5.1. For the second half we refer to Theorem 6.2 in Chapter 8 of [10]. $\square$

In view of Theorem 6.1, we may define a metric $d_{\dot{H}^1}(\cdot,\cdot)$ on M^k by letting $d_{\dot{H}^1}(\varphi,\psi)$ for $\varphi, \psi \in M^k$ be the length of the unique geodesic joining φ and ψ. Since $d(\varphi,\psi) = d(id, \varphi \circ \psi^{-1})$ whenever $\varphi, \psi \in M^k$ by right-invariance of the $\dot{H}^1$-metric, Corollary 5.2 yields

$$d_{\dot{H}^1}(\varphi,\psi) = d_{\dot{H}^1}(id, \varphi \circ \psi^{-1}) = \arccos\left(\int_{\mathbb{S}} \sqrt{(\varphi \circ \psi^{-1})_x} \mathrm{d}x\right). \tag{6.1}$$

A change of variables gives

$$d_{\dot{H}^1}(\varphi,\psi) = \arccos\left(\int_{\mathbb{S}} \sqrt{\varphi_x \psi_x} \mathrm{d}x\right). \tag{6.2}$$

We will write $(M^k, d_{\dot{H}^1}(\cdot,\cdot))$ when we consider M^k as a metric space endowed with this metric. An interesting global property of the manifold M^k is its diameter defined as

$$\mathrm{diam}(M^k) = \sup\left\{d_{\dot{H}^1}(\psi_0,\psi_1) \mid \psi_0, \psi_1 \in M^k\right\}.$$

Theorem 6.2. *The diameter of the space $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S}) \simeq M^k$ endowed with the $\dot{H}^1$ right-invariant metric is $\frac{\pi}{2}$.*

Proof. By (6.1) we have

$$\mathrm{diam}(M^k) = \sup\left\{d_{\dot{H}^1}(id,\psi) \mid \psi \in M^k\right\} = \arccos\left(\inf_{\psi \in M^k} \int_{\mathbb{S}} \sqrt{\psi_x} \mathrm{d}x\right). \tag{6.3}$$

In view of (2.1) the minimizing problem

$$\mathcal{I} = \inf_{\psi \in M^k} \int_{\mathbb{S}} \sqrt{\psi_x} \mathrm{d}x,$$

can be reformulated as

$$\mathcal{I} = \inf\left\{\int_{\mathbb{S}} \sqrt{1+f} \mathrm{d}x \,\middle|\, f \in H^{k-1}(\mathbb{S}),\ f > -1,\ \int_{\mathbb{S}} f \mathrm{d}x = 0\right\}.$$

Define a sequence $\{f_n\}_{n\geq 1}$ of functions $\mathbb{S} \simeq [0, 1) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} -1 + \dfrac{1}{n} & 0 \leq x < 1 - \dfrac{1}{n}, \\ n\left(1 - \dfrac{1}{n}\right)^2 & 1 - \dfrac{1}{n} \leq x < 1. \end{cases}$$

The functions f_n satisfy

$$f_n > -1 \quad \text{and} \quad \int_{\mathbb{S}} f_n \mathrm{d}x = 0, \quad n \geq 1,$$

and

$$\int_{\mathbb{S}} \sqrt{1 + f_n} \mathrm{d}x = \left(1 - \frac{1}{n}\right)\sqrt{\frac{1}{n}} + \frac{1}{n}\sqrt{1 + n\left(1 - \frac{1}{n}\right)^2} \to 0 \quad \text{as } n \to \infty.$$

Approximating the f_n's by smooth functions $\mathbb{S} \to \mathbb{R}$ with the same properties we infer that $\mathcal{I} = 0$, and so, by (6.3),

$$\operatorname{diam}(M^k) = \frac{\pi}{2}. \quad \square$$

We would also like to investigate the stability of the geodesic flow. The next result states exactly the rate at which the geodesics spread apart.

Theorem 6.3. *Let u_0, v_0 be two elements of $T_{id}M^k$ with $|u_0|_{id} = |v_0|_{id} = 1$. Consider the geodesics $\varphi : [0, T^*(u_0)) \to M^k$ and $\psi : [0, T^*(v_0)) \to M^k$ such that $\varphi(0) = \psi(0) = id$, $\varphi_t(0) = u_0$, and $\psi_t(0) = v_0$. For $t \in [0, T^*(u_0))$ and $s \in [0, T^*(v_0))$, it holds that*

$$d_{\dot{H}^1}(\varphi(t), \psi(s)) = \arccos\left(\cos t \cos s + \frac{2 - |u_0 - v_0|_{id}^2}{2} \sin t \sin s\right).$$

Proof. We compute, employing (2.7) and (6.2),

$$\begin{aligned} d_{\dot{H}^1}(\varphi(t), \psi(s)) &= \int_{\mathbb{S}} \sqrt{\varphi_x(t,x)\psi_x(s,x)} \mathrm{d}x \\ &= \int_{\mathbb{S}} \left(\cos t + \frac{u_{0x}}{2}\sin t\right)\left(\cos s + \frac{v_{0x}}{2}\sin s\right) \mathrm{d}x \\ &= \cos t \cos s + \langle u_0, v_0 \rangle_{id} \sin t \sin s. \end{aligned} \tag{6.4}$$

The identity

$$\langle u_0 - v_0, u_0 - v_0 \rangle_{id} = |u_0|_{id}^2 - 2\langle u_0, v_0 \rangle_{id} + |v_0|_{id}^2$$

gives the stated result. $\square$

7. Geodesic flow on a sphere

In view of the previous results it is clear that $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S}) \simeq M^k$ shares many properties of a convex subset of a unit sphere, e.g. the curvature is constant equal to one and the exponential map provides global normal coordinates. By constructing an explicit isometry from M^k to an open subset of the L^2-sphere

$$\mathcal{S}^{k-1} = \{f \in H^{k-1}(\mathbb{S}) \mid \|f\|_{L^2} = 1\},$$

we will see that this intuition can be firmly established. In finite dimensions any Riemannian manifold with constant positive curvature is locally isometric to a sphere (cf. [11]). It is therefore not the existence of an isometry that is most surprising, but the simplicity of it.

Note that $\mathcal{S}^{k-1} = F^{-1}(1)$ where $F : H^{k-1}(\mathbb{S}) \to \mathbb{R}$ is the smooth functional $u \mapsto \|u\|_{L^2(\mathbb{S})}^2$. Since $T_f F = 2\langle f, \cdot\rangle_{L^2}$ is surjective with a splitting kernel at any point $f \in \mathcal{S}^{k-1}$, we deduce that $\mathcal{S}^{k-1}$ is a closed submanifold of

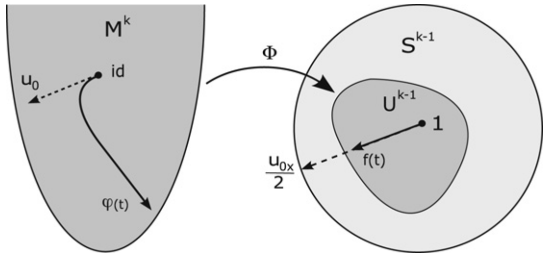

Fig. 1. The map $\Phi : M^k \to \mathcal{U}^{k-1}$ given by $\varphi \mapsto \sqrt{\varphi_x}$ is a diffeomorphism and an isometry.

$H^{k-1}(\mathbb{S})$ (cf. [10]). We endow $\mathcal{S}^{k-1}$ with the induced manifold structure and define a weak Riemannian metric on $\mathcal{S}^{k-1}$ by

$$\langle \xi, \eta \rangle_f = \langle \xi, \eta \rangle_{L^2},$$

for $f \in \mathcal{S}^{k-1}$ and $\xi, \eta \in T_f\mathcal{S}^{k-1} \simeq \{\zeta \in H^{k-1}(\mathbb{S}) \mid \langle \zeta, f \rangle_{L^2} = 0\}$. We also define a metric $d_{L^2}(\cdot, \cdot)$ on $\mathcal{S}^{k-1}$ by the formula

$$d_{L^2}(f, g) = \arccos(\langle f, g \rangle_{L^2}),$$

that is, $d_{L^2}(f, g)$ is the L^2-angle between f and g. Let us check that $d_{L^2}(\cdot, \cdot)$ satisfies the triangle inequality. For $f, g, h \in \mathcal{S}^{k-1}$ we introduce L^2-perpendicular vectors $e_1, e_2, e_3 \in \mathcal{S}^{k-1}$ such that $f = e_1$, $g = a_1e_1 + a_2e_2$, and $h = b_1e_1+b_2e_2+b_3e_3$ for some constants $a_1, a_2, b_1, b_2, b_3 \in \mathbb{R}$. The triangle inequality $d_{L^2}(f, h) \le d_{L^2}(f, g)+d_{L^2}(g, h)$ amounts to

$$\arccos(b_1) \le \arccos(a_1) + \arccos(a_1b_1 + a_2b_2),$$

whenever $a_1^2 + a_2^2 = b_1^2 + b_2^2 + b_3^2 = 1$. But this is just the triangle inequality for the geodesic triangle on the unit sphere in $\mathbb{R}^3$ with corners at $(1, 0, 0)$, $(a_1, a_2, 0)$, and (b_1, b_2, b_3). We deduce that $d_{L^2}(\cdot, \cdot)$ indeed is a metric on $\mathcal{S}^{k-1}$.

Let $\mathcal{U}^{k-1}$ be the open subset of $\mathcal{S}^{k-1}$ given by

$$\mathcal{U}^{k-1} = \{f \in \mathcal{S}^{k-1} \mid f > 0 \text{ on } \mathbb{S}\}.$$

We equip $\mathcal{U}^{k-1}$ with the manifold structure inherited from $\mathcal{S}^{k-1}$. Note that the topology induced by $d_{L^2}(\cdot, \cdot)$ on $\mathcal{S}^{k-1}$ is strictly weaker than the topology defined by the manifold structure. In particular, $\mathcal{U}^{k-1}$ is not open in the metric space $(\mathcal{S}^{k-1}, d_{L^2}(\cdot, \cdot))$. This is an effect of the fact that we are considering a *weak* Riemannian metric.

Theorem 7.1. *The map*

$$\Phi : \varphi \mapsto f = \sqrt{\varphi_x}$$

is a diffeomorphism $M^k \to \mathcal{U}^{k-1}$ and an isometry between the weak Riemannian manifolds $(M^k, \langle \cdot, \cdot \rangle_{\dot{H}^1})$ and $(\mathcal{U}^{k-1}, \langle \cdot, \cdot \rangle_{L^2})$, that is,

$$\langle T_\varphi\Phi(U), T_\varphi\Phi(V) \rangle_{L^2} = \langle U, V \rangle_\varphi,$$

for any $\varphi \in M^k$ and tangent vectors $U, V \in T_\varphi M^k$ (see Fig. 1). In particular, Φ is an isometry from the metric space $(M^k, d_{\dot{H}^1}(\cdot, \cdot))$ to $(\mathcal{U}^{k-1}, d_{L^2}(\cdot, \cdot))$.

Proof. To see that Φ is bijective we construct its inverse. Put, for $f \in \mathcal{U}^{k-1}$,

$$\varphi(x) = \int_0^x f^2(y)\mathrm{d}y.$$

Then $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi_x = f^2 \in H^{k-1}(\mathbb{S})$, and $\varphi_x > 0$. Hence $\varphi \in M^k$. As $\Phi(\varphi) = f$ we deduce that Φ is bijective. Since both Φ and its inverse are easily seen to be smooth, it follows that Φ is a diffeomorphism $M^k \to \mathcal{U}^{k-1}$.

Using that

$$T_\varphi\Phi(U) = \frac{U_x}{2\sqrt{\varphi_x}},$$

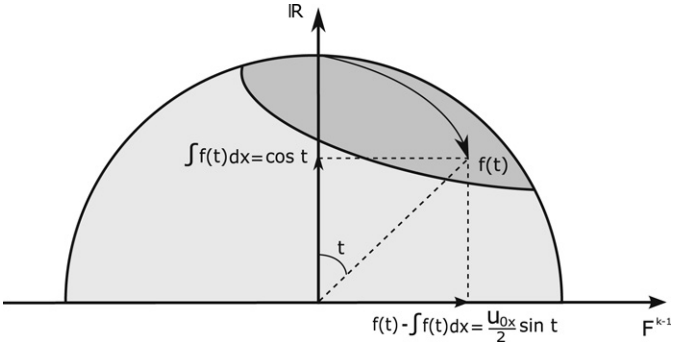

Fig. 2. A geodesic $t \mapsto \varphi(t)$ in M^k with $\varphi(0) = id$ and $\varphi_t(0) = u_0$ is mapped by Φ to a geodesic $t \mapsto f(t)$ in $\mathcal{U}^{k-1}$ with $f(0) = 1$ and $f_t(0) = \frac{u_{0x}}{2}$.

we get, for $U, V \in T_\varphi M^k$,

$$\langle T_\varphi \Phi(U), T_\varphi \Phi(V)\rangle_{L^2} = \frac{1}{4}\int_{\mathbb{S}} \frac{U_x V_x}{\varphi_x}\mathrm{d}x.$$

But this equals

$$\langle U, V\rangle_\varphi = \frac{1}{4}\int_{\mathbb{S}} \frac{U_x V_x}{\varphi_x}\mathrm{d}x,$$

showing that Φ is an isometry. The fact that Φ satisfies

$$d_{\dot{H}^1}(\varphi, \psi) = d_{L^2}(\Phi(\varphi), \Phi(\psi)),$$

can either be seen as a consequence of the previous statement or deduced directly from expression (6.2) for $d_{\dot{H}^1}(\varphi, \psi)$. □

Several of the results obtained for M^k in the previous sections can now be interpreted on the sphere.

Group structure. The composition on $M^k = \{\varphi \in \mathcal{D}^k(\mathbb{S}) \mid \varphi(0) = 0\}$ is transferred to a multiplication $\triangle$ on $\mathcal{U}^{k-1}$ such that $\Phi(\varphi \circ \psi) = \Phi(\varphi) \triangle \Phi(\psi)$ for $\varphi, \psi \in M^k$. With $f = \Phi(\varphi)$, $g = \Phi(\psi)$, we get

$$f \triangle g = f \circ \psi \cdot g = \left(x \mapsto f\left(\int_0^x g^2(y)\mathrm{d}y\right) g(x)\right),$$

and $g^{-1} = \Phi(\psi^{-1}) = \frac{1}{g\circ\psi^{-1}}$. We conclude that the right multiplication operators $R_g(f) = f \triangle g$ preserve the L^2-metric.

Geodesics. The geodesic $f : J \to \mathcal{U}^{k-1}$ with $f(0) \equiv 1$ and $f_t(0) = \frac{u_{0x}}{2} \in T_1\mathcal{S}^{k-1} = \mathbf{F}^{k-1}$, $\|\frac{u_{0x}}{2}\|_{L^2} = 1$, is given by $\Phi \circ \varphi$ where φ is the geodesic in M^k with $\varphi(0) = id$ and $\varphi_t(0) = u_0$. By (2.7) we have (see Fig. 2)

$$f(t) = \cos t + \frac{u_{0x}}{2}\sin t.$$

Note that $TR_g(\eta) = \eta \circ \psi \cdot g$ for $\eta \in T_g\mathcal{S}^{k-1}$ and $\psi(x) = \int_0^x g^2(y)\mathrm{d}y$. Therefore, the geodesic starting at g with initial velocity $TR_g(\frac{u_{0x}}{2}) = \frac{u_{0x}}{2} \circ \psi \cdot g \in T_{g^{-1}}\mathcal{S}^{k-1}$ is given by

$$t \mapsto f(t) \triangle g = \left(x \mapsto \left(\cos t + \frac{u_{0x}\left(\int_0^x g^2(y)\mathrm{d}y\right)}{2}\sin t\right) g(x)\right).$$

This gives explicit formulas for all geodesics in $\mathcal{U}^{k-1}$.

Parallel translation. Let f and φ be as before and let $\xi : J \to T\mathcal{U}^{k-1}$ be the parallel translation of a vector $\frac{v_{0x}}{2} \in T_1\mathcal{S}^{k-1}$ along f. Then $\xi(t) = T\Phi \circ V(t)$ where $V : J \to TM^k$ is the parallel translation of v_0 along φ.

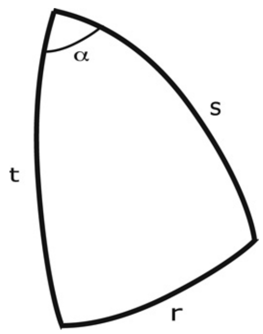

Fig. 3. The sides of a geodesic triangle on the unit sphere are related as $\cos r = \cos t \cos s + \cos\alpha \sin t \sin s$.

From Lemma 4.2 we infer that

$$\xi(t,x) = \frac{V_x(t,x)}{2\sqrt{\varphi_x}} = -\frac{1}{4}\langle u_0, v_0\rangle_{id}\left(\sin t + \frac{u_{0x}}{2}(1-\cos t)\right) + \frac{v_{0x}}{2}.$$

More generally, the parallel translation along the geodesic starting at a point $\Phi(\psi) = g \in \mathcal{U}^{k-1}$ with initial velocity $TR_g(\frac{u_{0x}}{2}) = \frac{u_{0x}}{2} \circ \psi \cdot g \in T_{g^{-1}}\mathcal{S}^{k-1}$ can be obtained as $TR_g \circ \xi = \xi \circ \psi \cdot g$.

Jacobi fields. With $f(t) = \cos t + \frac{u_{0x}}{2}\sin t$ as above, let w_{0x} be an element of $T_1\mathcal{S}^{k-1}$ with decomposition $w_{0x} = c_0 v_{0x} + c_1 u_{0x}$ for some $c_0, c_1 \in \mathbb{R}$, and $v_{0x} \in \mathcal{S}^{k-1} L^2$-orthogonal to u_{0x}. The unique Jacobi field $t \mapsto \eta(t)$ along f with $\eta(0) = 0$ and $\nabla_{f_t}\eta(0) = \frac{w_{0x}}{2}$ is given by

$$\eta(t) = c_0 t f_t(t) + c_1 \frac{v_{0x}}{2}\sin t.$$

Diameter. The fact that $\operatorname{diam}(M^k) \le \frac{\pi}{2}$ means that the L^2-angle between two vectors $f, g \in \mathcal{U}^{k-1}$ is always less than $\frac{\pi}{2}$. This is obvious since

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{S}} fg\,\mathrm{d}x > 0$$

whenever $f, g > 0$. On the other hand, the equality $\operatorname{diam}(M^k) = \frac{\pi}{2}$ implies the existence of functions $f, g \in \mathcal{U}^{k-1}$ with $\langle f, g\rangle_{L^2}$ arbitrarily close to zero.

Geodesic triangle. Eq. (6.4) is the well-known formula for the lengths of the sides of a geodesic triangle on a sphere: if two sides, separated by an angle α, have lengths t respectively s, then the length r of the third side satisfies (see Fig. 3)

$$\cos r = \cos t \cos s + \cos\alpha \sin t \sin s.$$

Killing fields. Let $t \mapsto \psi(t)$ be a curve in M^k with $\psi(0) = id$ and $\psi_t(0) = v \in T_{id}M^k$. For each t the map $\varphi \mapsto \varphi \circ \psi(t)$ is an isometry $M^k \to M^k$. Hence the vector field

$$X(\varphi) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \varphi \circ \psi(t) = v\varphi_x$$

is a Killing field on M^k. The isometry $\Phi : M^k \to \mathcal{U}^{k-1}$ transforms X into the vector field ξ on $\mathcal{U}^{k-1}$ given by

$$\xi(\sqrt{\varphi_x}) = T_\varphi\Phi(X(\varphi)) = \frac{(\varphi_x v)_x}{2\sqrt{\varphi_x}}.$$

Therefore we have shown that, for any fixed $v \in \mathbf{E}^k$ the vector field

$$\xi : f \in \mathcal{U}^{k-1} \mapsto f_x v + \frac{1}{2} f v_x \in T_f\mathcal{S}^{k-1}.$$

is a killing field on $\mathcal{U}^{k-1}$.

7.1. Parallel 1,1-tensors

To illustrate that the isometry Φ can also give new information about the quotient space $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$, we consider the following problem.

Problem 7.2. Determine the φ-parallel 1,1-tensor fields along a geodesic φ in M^k.

If $\varphi : J \to M^k$ is a geodesic, a 1,1-tensor along φ is a smooth map $L : J \to \mathcal{L}(TM^k, TM^k)$ such that $L(t) \in \mathcal{L}(T_{\varphi(t)}M^k, T_{\varphi(t)}M^k)$ for each t. Here $\mathcal{L}(TM^k, TM^k)$ denotes the vector bundle over M^k with the space $\mathcal{L}(T_\psi M^k, T_\psi M^k)$ of linear continuous maps $T_\psi M^k \to T_\psi M^k$ as the fiber over $\psi \in M^k$. By definition the tensor field L is φ-parallel if $\nabla_{\varphi_t} L \equiv 0$.

Let us point out that φ-parallel 1,1-tensors could prove relevant for the geometric study of Lax pairs for the Hunter–Saxton equation. Indeed, for a vector field X along φ we have by definition

$$(\nabla_{\varphi_t} L)X = \nabla_{\varphi_t}(L(X)) - L(\nabla_{\varphi_t} X),$$

so that in the chart M^k

$$(\nabla_{\varphi_t} L)X = L_t(X) - \Gamma(\varphi, L(X), \varphi_t) + L(\Gamma(\varphi, X, \varphi_t)).$$

Letting $B : J \to \mathcal{L}(\mathbf{E}^k, \mathbf{E}^k)$ be the 1,1-tensor along φ given by $B(t) = -\Gamma(\varphi, \cdot, \varphi_t)$, we see that

$$\nabla_{\varphi_t} L = L_t - [L, B].$$

Thus $\nabla_{\varphi_t} L \equiv 0$ if and only if L and B satisfy the Lax equation $L_t = [L, B]$. Now it appears difficult to determine $L : J \to L(\mathbf{E}^k, \mathbf{E}^k)$ from the equation $L_t = [L, B]$ when $B = -\Gamma(\varphi, \cdot, \varphi_t)$. However, we will see that by means of Φ we can reduce Problem 7.2 to a trivial one.

Consider the stereographic projection from the south pole $\sigma : \mathcal{U}^{k-1} \to \mathbf{F}^{k-1}$ given by

$$\sigma(f) = \frac{f - \int_{\mathbb{S}} f\,\mathrm{d}x}{1 + \int_{\mathbb{S}} f\,\mathrm{d}x}.$$

It is easy to check that σ is a diffeomorphism from $\mathcal{U}^{k-1}$ onto

$$\mathcal{V}^{k-1} = \{\alpha \in \mathbf{F}^{k-1} \mid 2\alpha > |\alpha|_{L^2}^2 - 1 \text{ on } \mathbb{S}\},$$

with inverse

$$\sigma^{-1} : \alpha \mapsto \frac{2\alpha - |\alpha|_{L^2}^2 + 1}{|\alpha|_{L^2}^2 + 1}.$$

Moreover, using that

$$T_f\sigma(\xi) = \frac{\xi\left(1 + \int_{\mathbb{S}} f\,\mathrm{d}x\right) - (1 + f)\int_{\mathbb{S}} \xi\,\mathrm{d}x}{\left(1 + \int_{\mathbb{S}} f\,\mathrm{d}x\right)^2}.$$

we conclude that σ is an isometry from $(\mathcal{U}^{k-1}, \langle\cdot,\cdot\rangle_{L^2})$ to $(\mathcal{V}^{k-1}, \frac{4}{(|\alpha|_{L^2}^2+1)^2}\langle\cdot,\cdot\rangle_{L^2})$, that is, for any $f \in \mathcal{U}^{k-1}$ with $\sigma(f) = \alpha$ and $\xi, \eta \in T_f\mathcal{S}^{k-1}$, it holds that

$$\langle\xi, \eta\rangle_{L^2} = \frac{4}{(|\alpha|_{L^2}^2 + 1)^2}\langle T_f\sigma(\xi), T_f\sigma(\eta)\rangle_{L^2}.$$

Thus $\mathcal{V}^{k-1}$ provides a global chart for $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}^k(\mathbb{S})$.

The Christoffel map $\bar{\Gamma} : \mathcal{V}^{k-1} \times \mathbf{F}^{k-1} \times \mathbf{F}^{k-1} \to \mathbf{F}^{k-1}$ for the metric spray on $\mathcal{V}^{k-1}$ is characterized as usual by the local formula (cf. [10])

$$-2\langle\bar{\Gamma}(\alpha, U, V), g(\alpha)W\rangle_{L^2} = \langle Dg(\alpha)\cdot U\cdot W, V\rangle_{L^2} + \langle Dg(\alpha)\cdot V\cdot W, U\rangle_{L^2} - \langle Dg(\alpha)\cdot W\cdot V, U\rangle_{L^2},$$

where, for each $\alpha \in \mathcal{V}^{k-1}$, the local representative $g(\alpha) : \mathbf{F}^{k-1} \to \mathbf{F}^{k-1}$ of the metric is the multiplication operator

$$g(\alpha) : U \mapsto \frac{4U}{(|\alpha|_{L^2}^2 + 1)^2}.$$

Since, for $U, V \in \mathbf{F}^{k-1}$,

$$Dg(\alpha) \cdot V \cdot U = \frac{-4\langle \alpha, V\rangle_{L^2} g(\alpha)U}{|\alpha|_{L^2}^2 + 1},$$

we find that

$$\bar{\Gamma}(\alpha, U, V) = 2\frac{\langle \alpha, U\rangle_{L^2} V + \langle \alpha, V\rangle_{L^2} U - \langle U, V\rangle_{L^2}\alpha}{|\alpha|_{L^2}^2 + 1}.$$

Now let φ be a geodesic in M^k. By right-invariance we may assume that φ starts at id with some initial velocity $\varphi_t(0) = u_0 \in T_{id}M^k$. In $\mathcal{V}^{k-1}$, φ corresponds to the geodesic $\sigma \circ \Phi \circ \varphi = \alpha : J \mapsto \mathcal{V}^{k-1}$ given by

$$\alpha : t \mapsto \frac{\sin t}{1 + \cos t}\alpha_0,$$

where $\alpha_0 = \frac{u_{0x}}{2}$. Since $\alpha_t(t) = \frac{1}{1+\cos t}\alpha_0$, we see that $U \mapsto \bar{\Gamma}(\alpha, U, \alpha_t) : \mathbf{F}^{k-1} \to \mathbf{F}^{k-1}$ is just the multiplication operator

$$U \mapsto \frac{\sin t}{1 + \cos t}U.$$

Hence, if $L : J \to \mathcal{L}(\mathbf{F}^{k-1}, \mathbf{F}^{k-1})$ is a 1,1-tensor field along α, we have

$$(\nabla_{\alpha_t} L)(X) = L_t X - \Gamma(\varphi, L(X), \varphi_t) + L(\Gamma(\varphi, X, \varphi_t)) = L_t X,$$

so that the equation $\nabla_{\varphi_t}\bar{L} \equiv 0$ implies that $L \equiv L_0$ is a constant linear operator $\mathbf{F}^{k-1} \to \mathbf{F}^{k-1}$. Therefore, in the chart $\mathcal{V}^{k-1}$ the computation of the 1,1-tensors parallel along a geodesic is trivial. Since we know the change-of-chart map $\sigma \circ \Phi : M^k \to \mathcal{V}^{k-1}$ and its inverse explicitly, it is now straightforward to obtain an explicit formula for L in the chart M^k. This solves Problem 7.2.

8. Conclusions and remarks

The material presented in this paper shows that Eq. (1.1) can be viewed as the Euler equation for the geodesic flow on a sphere. The simplicity of this geometric interpretation is related to the fact that solutions of (1.1) in Lagrangian coordinates exhibit very simple time dependence. It is also reflected in several other properties of the space $\mathrm{Rot}(\mathbb{S})\backslash\mathcal{D}(\mathbb{S})$ endowed with the $\dot{H}^1$ right-invariant metric: its diameter is $\frac{\pi}{2}$, the curvature is positive and constant, there exist no conjugate points along any geodesics, and there exists a unique, globally length-minimizing, geodesic joining any two points of the manifold. It is perhaps surprising that so much can be said about an infinite-dimensional diffeomorphism group. For example, while facts such as the existence of a globally length-minimizing geodesic between any two points of a manifold can sometimes be obtained in the finite-dimensional setting by means of the Hopf–Rinow theorem, the existence of minimal geodesics in infinite dimensions is usually very difficult to ascertain. The Hopf–Rinow theorem is not available in the infinite-dimensional case due to the lack of local compactness, and it is indeed easy to give an example of an infinite-dimensional ellipsoid on which there is no shortest geodesic joining two antipodal points (see [10]).

The most widely studied example of an Euler equation describing the geodesic motion on a diffeomorphism group are the Euler equations of fluid dynamics modeling the motion of non-viscous, homogeneous, and incompressible fluid moving in a bounded smooth domain $\mathcal{M} \subset \mathbb{R}^n$. It was first discovered in [1], and later put on a rigorous mathematical foundation in [5], that endowing the group $\mathcal{D}_\mu(\mathcal{M})$ of volume-preserving diffeomorphisms of $\mathcal{M}$ with the L^2 right-invariant metric, the geodesics are described by the classical Euler equations. Since then much effort has been put into understanding the geodesic flow on $\mathcal{D}_\mu(\mathcal{M})$ in order to draw conclusions about the fluid motion. Also other well-known nonlinear wave equations have been found to arise as Euler equations for the geodesic flow on diffeomorphism groups endowed with invariant metrics. For example, the Euler equation describing the geodesics on the Virasoro group (a one-dimensional extension of the diffeomorphism group of the circle) equipped with the L^2 right-invariant metric, is the well-known Korteweg–de Vries equation [15], while the H^1 right-invariant metric yields the Camassa–Holm equation [14].

In all these cases, however, the success of the geometric approach to the study of e.g. stability questions has been limited because the underlying manifolds of diffeomorphisms have turned out to be very intricate objects. The results in this paper show that the Hunter–Saxton equation provides an example of an Euler equation on an infinite-dimensional diffeomorphism group which has enough structure to be interesting, but is still simple enough to be tractable to analysis. Hopefully the insight gained from a study of such an intermediate example will prove useful also in the investigation of the more complicated instances.

Finally, we mention some questions raised by the present work:

- Can the sphere interpretation be used to extend solutions of the Hunter–Saxton equation beyond breaking time? Whereas all geodesics in the diffeomorphism group break in finite time, nothing hinders us to extend the corresponding geodesics on the sphere indefinitely.
- Does the constant curvature of the $\dot{H}^1$ right-invariant metric tell us anything about the full H^1 metric on the diffeomorphism group?
- Is there a connection between the integrability of the Hunter–Saxton equation and the fact that it describes geodesic motion on a sphere?

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References

[1] V. Arnold, Sur la géometrie différentielle des groupes de Lie de dimension infinie et ses application à l'hydrodynamique des fluides parfaits, Ann. Inst. Fourier (Grenoble) 16 (1966) 319–361.
[2] A. Bressan, A. Constantin, Global solutions of the Hunter–Saxton equation, SIAM J. Math. Anal. 37 (2005) 996–1026.
[3] R. Camassa, D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 1661–1664.
[4] A. Constantin, B. Kolev, On the geometric approach to the motion of inertial mechanical systems, J. Phys. A 35 (2002) R51–R79.
[5] D. Ebin, J. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, Ann. Math. 92 (1970) 102–163.
[6] R. Hamilton, The inverse function theorem of Nash and Moser, Bull. Amer. Math. Soc. 7 (1982) 65–222.
[7] J.K. Hunter, R. Saxton, Dynamics of director fields, SIAM J. Appl. Math. 51 (1991) 1498–1521.
[8] J.K. Hunter, Y. Zheng, On a completely integrable nonlinear hyperbolic variational equation, Physica D 79 (1994) 361–386.
[9] B. Khesin, G. Misiołek, Euler equations on homogeneous spaces and Virasoro orbits, Adv. Math. 176 (2003) 116–144.
[10] S. Lang, Differential and Riemannian Manifolds, 3rd ed., Springer-Verlag, New York, 1995.
[11] J.M. Lee, Riemannian Manifolds, Springer-Verlag, New York, 1997.
[12] J. Lenells, The Hunter–Saxton equation: A geometric approach, Preprints in Mathematical Sciences 2006:5, Lund University.
[13] G. Misiołek, Gerard Conjugate points in the Bott–Virasoro group and the KdV equation, Proc. Amer. Math. Soc. 125 (1997) 935–940.
[14] G. Misiołek, A shallow water equation as a geodesic flow on the Bott–Virasoro group, J. Geom. Phys. 24 (1998) 203–208.
[15] V. Ovsienko, B. Khesin, The (super) KdV equation as an Euler equation, Funct. Anal. Appl. 21 (4) (1987) 81–82.
[16] S. Shkoller, Geometry and curvature of diffeomorphism groups with H^1 metric and mean hydrodynamics, J. Funct. Anal. 160 (1998) 337–365.